

# ON THE UNITARY SUBGROUP OF MODULAR GROUP ALGEBRAS

VICTOR BOVDI, TIBOR ROZGONYI<sup>1</sup>

ABSTRACT. In this note we investigate the structure of the group of  $\sigma$ -unitary units in some noncommutative modular group algebras  $KG$ , where  $\sigma$  is a non-classical ring involution of  $KG$ .

Let  $K$  be a field of characteristic  $p$  and  $KG$  be the group algebra of the finite  $p$ -group  $G$  over  $K$ . The group of units of  $KG$  is denoted by  $U(KG)$  and its subgroup

$$V(KG) = \left\{ \sum_{g \in G} \alpha_g g \in U(KG) \mid \sum_{g \in G} \alpha_g = 1 \right\}$$

is called the group of normalized units. Let  $\sigma$  be an anti-automorphism of order two of the group  $G$ . If  $x = \sum_{g \in G} \alpha_g g$  is an element of the algebra  $KG$ , then  $u^\sigma$  denote the element  $\sum_{g \in G} \alpha_g g^\sigma$ . The map  $u \mapsto u^\sigma$  is called  $\sigma$ -involution and the following relations hold:

1.  $(u + v)^\sigma = u^\sigma + v^\sigma$ ;
2.  $(uv)^\sigma = v^\sigma u^\sigma$ ;
3.  $(u^\sigma)^\sigma = u$ , for all  $u, v \in KG$ .

We give example such involutions.

1. Let  $\sigma(g) = g^{-1}$  for all  $g \in G$ . It is obvious that  $x \mapsto x^\sigma$  is a classical involution of  $KG$  and  $x^\sigma$  is denoted by  $x^*$ .
2. Let  $G$  be a finite 2-group and  $C$  be the center of  $G$  such that  $G/C \cong C_2 \times C_2$  and the commutator subgroup  $G' = \langle e \mid e^2 = 1 \rangle$  is of order two. Then the mapping  $\odot : G \mapsto G$ , defined by

$$g^\odot = \begin{cases} g, & \text{if } g \in C, \\ ge, & \text{if } g \notin C \end{cases}$$

is an anti-automorphism of order two. If  $x = \sum_{g \in G} \alpha_g g \in KG$ , then  $u \mapsto u^\odot = \sum_{g \in G} \alpha_g g^\odot$  is an involution.

An element  $u \in V(KG)$  is called  $\sigma$ -unitary if  $u^{-1} = u^\sigma$ . The set of all unitary elements of the group  $V(KG)$  is a subgroup. We shall denote this subgroup by  $V_\sigma(KG)$  and refer to  $V_\sigma(KG)$  as the unitary subgroup of  $V(KG)$ . It is clear that  $G$  is a subgroup of  $V_\sigma(KG)$ .

A.A. Bovdi and A.A. Szakács in [1] describe the group  $V_*(KG)$  for an arbitrary finite abelian  $p$ -group  $G$  and a finite field  $K$  of characteristic  $p$ . A.A. Szakács

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calculated the unitary subgroup of commutative group algebras in [4-6]. In case of a non-classical involution  $\sigma$  nothing is known of  $V_\sigma(KG)$ .

It is an open and important problem to investigate  $V_\sigma(KG)$  in the non-abelian case. The following results we obtain may be considered the first ones in this direction.

**Theorem 1.** *Let  $G$  be a finite 2-group which contains an abelian normal subgroup  $A$  of index two. Suppose that there exists an element  $b \in G \setminus A$  of order 4 such that  $b^{-1}ab = a^{-1}$  for all  $a \in A$ . Then the unitary subgroup  $V_*(\mathbb{Z}_2G)$  is the semidirect product of  $G$  and a normal subgroup  $H$ .*

*The subgroup  $H$  is the semidirect product of the normal elementary abelian 2-group*

$$W = \{1 + (1 + b^2)zb \mid z \in \mathbb{Z}_2A\}$$

*and the abelian subgroup  $L$ , where  $V_*(\mathbb{Z}_2A) = A \times L$ .*

*The abelian group  $W$  is the direct product of  $\frac{1}{2}|A|$  copies of the additive group of field  $\mathbb{Z}_2$ .*

**Proof.** It is well known [1] that

$$V_*(\mathbb{Z}_2A) = A \times L. \quad (1)$$

Suppose now that  $G = \langle A, b \rangle$ , and  $b^{-1}ab = a^{-1}$  for all  $a \in A$ . Then every element of  $V_*(\mathbb{Z}_2G)$  has a unique representation of form

$$x = x_1 + x_2b \quad (x_i \in \mathbb{Z}_2A).$$

Let  $\chi(x_i)$  be the sum of the coefficients of the element  $x_i$ .

Let  $J(A)$  denote the ideal of the ring  $\mathbb{Z}_2G$  generated by the elements of form  $h - 1$  with  $h \in A$ . Since  $\mathbb{Z}_2G/J(A) \cong \mathbb{Z}_2(G/A)$  and quotient group  $G/A$  is of order 2, it follows that element

$$\bar{x} = \chi(x_1) + \chi(x_2)\bar{b} \quad (\bar{b} = bA)$$

is trivial. Hence, one of the elements  $\chi(x_1)$  or  $\chi(x_2)$  equals 1 and the other is zero. Since  $G \subseteq V_*(\mathbb{Z}_2G)$ , element  $xb \in V_*(\mathbb{Z}_2G)$  and  $x$  or  $xb$  have the form

$$x_i(1 + x_i^{-1}x_jb), \quad (i, j \in \{1, 2\}, i \neq j).$$

Indeed, if  $\chi(x_i) = 1$ , then  $x_i$  is a unit.

Suppose that  $x = x_1(1 + x_2b) \in V_*(\mathbb{Z}_2G)$  and  $x_1 \in V(\mathbb{Z}_2A)$  and  $\chi(x_2) = 0$ . Then

$$x^* = (1 + b^{-1}x_2^*)x_1^* \quad \text{and} \quad xx^* = 1.$$

It follows that

$$\begin{cases} x_1x_1^*(1 + x_2x_2^*) = 1, \\ x_2(1 + b^2) = 0. \end{cases} \quad (2)$$

Then by Proposition of [2] we have

It is obvious that

$$x_2 x_2^* = 2(1 + b^2) z z^* = 0.$$

From (2) we have  $x_1 x_1^* = 1$  and element  $x_1$  is unitary. Therefore  $x_1 \in V_*(\mathbb{Z}_2 A)$ .

Let  $g_1, \dots, g_t$  be the representatives of the distinct cosets of  $A$  modulo  $\langle b^2 \rangle$  and

$$W_{g_i} = \{1 + (1 + b^2)\alpha g_i b \mid \alpha \in \mathbb{Z}_2\}.$$

It is clear that  $W_{g_i}$  is a group of order 2 and  $W = \prod_{i=1}^z W_{g_i}$  is a direct product. If  $w_i = 1 + (1 + b^2)g_i b$  and  $b g_i b^{-1} = g_i^{-1} = g_j b^k$  ( $k \geq 2$ ) then

$$b w_i b^{-1} = 1 + (1 + b^2)g_j b \in W \quad (3)$$

and for every  $x_1 \in V_*(\mathbb{Z}_2 A)$

$$x_1 w_i x_1^{-1} = 1 + (1 + b^2)x_1^2 g_i b \in W, \quad (4)$$

because

$$b x_1^{-1} = b x_1^* = x_1 b \quad (5)$$

Thus  $W$  is a normal subgroup of  $V_*(\mathbb{Z}_2 G)$  and  $W \cap L = 1$ , where  $V_*(\mathbb{Z}_2 A) = A \times L$ .

Let  $H$  be the subgroup generated by  $W$  and  $L$ . By (4) it follows that  $H$  is a semidirect product of normal subgroup  $W$  and the subgroup  $L$ .

Therefore we proved that  $V_*(\mathbb{Z}_2 G)$  is generated by  $G$  and  $H$ . From (3), (4) and (5) it follows that  $H$  is a normal subgroup of  $V_*(\mathbb{Z}_2 G)$ .

**Theorem 2.** *Let  $C$  be the center of a finite 2-group  $G$  and*

1.  *$G/C$  is a direct product of two groups  $\langle aC \rangle$  and  $\langle bC \rangle$  of order two;*
2. *the commutator subgroup  $G' = \langle e \rangle$  of group  $G$  is order 2;*

*Then the unitary subgroup*

$$V_{\odot}(\mathbb{Z}_2 G) = G \times T \times W,$$

*where  $W = \{1 + x_1 a + x_2 b + x_3 ab \mid x_i \in \mathbb{Z}_2 C(1 + e)\}$  is a central elementary abelian 2-group, the subgroup  $V(\mathbb{Z}_2 C)[2]$  of all elements of order 2 in  $V(\mathbb{Z}_2 C)$  is a direct product of group  $T$  and the subgroup  $C[2]$  of all elements of order 2 in  $C$ .*

**Proof.** Let  $a, b \in G$  and  $[a, b] = e$ . Then  $C = C_G(a, b)$  is the center of  $G$  and

$$G = C \cup Ca \cup Cb \cup Cab.$$

Every element of  $V_{\odot}(\mathbb{Z}_2 G)$  can be written in the form

$$x = x_0 + x_1 a + x_2 b + x_3 ab \quad (x_i \in \mathbb{Z}_2 C)$$

and elements  $x_i$  are central in  $KG$ . Then

$$x^{\odot} = x_0 + x_1 a e + x_2 b e + x_3 a b e \quad (x_i \in \mathbb{Z}_2 C)$$

and a simple calculation immediately shows that  $x \in V_{\odot}(\mathbb{Z}_2 G)$  if and only if

$$\begin{cases} x_0^2 + x_1^2 a^2 e + x_2^2 b^2 e + x_3^2 a^2 b^2 = 1, \\ (x_0 x_1 + x_2 x_3 b^2)(1 + e) = 0, \\ (x_0 x_2 + x_1 x_3 a^2)(1 + e) = 0, \\ (x_0 x_3 + x_1 x_2 a b)(1 + e) = 0. \end{cases} \quad (6)$$

Since  $\chi(x_i)$  equals 1 or 0 and  $G \subseteq V_{\odot}(\mathbb{Z}_2G)$  we may assume that  $\chi(x_0) = 1$ . By Propositions 2.7 [2] from (6), we obtain

$$x_0x_1 + x_2x_3b^2 = (1+e)r_1,$$

$$x_0x_2 + x_1x_3a^2 = (1+e)r_2,$$

$$x_0x_3 + x_1x_2 = (1+e)r_3,$$

for some  $r_i \in \mathbb{Z}_2C$ .

Multiplying the first equality by  $x_1a^2$  and the second equality by  $x_2b^2$  we have

$$x_0(x_1^2a^2 + x_2^2b^2) = (1+e)r_4,$$

$$x_0(x_1^2 + x_3^2b^2) = (1+e)r_5,$$

$$x_0(x_2^2 + x_3^2a^2) = (1+e)r_6,$$

for some  $r_i \in \mathbb{Z}_2C$ . Furthermore, Proposition 2.6 [2] assures that  $\chi$  is a ring homomorphism. Because  $\chi(x_0) = 1$ , then we have

$$\begin{cases} \chi(x_1) + \chi(x_2) = 0, \\ \chi(x_1) + \chi(x_3) = 0, \\ \chi(x_2) + \chi(x_3) = 0. \end{cases}$$

Clearly, if  $x \in V(\mathbb{Z}_2G)$  then  $\chi(x) = \chi(x_0) + \chi(x_1) + \chi(x_2) + \chi(x_3) = 1$  and therefore it follows that  $\chi(x_1) = \chi(x_2) = \chi(x_3) = 0$ .

Since  $x_0$  is a unit,  $x$  can be written as

$$x = x_0(1 + y_1a + y_2b + y_3ab),$$

where  $\chi(y_i) = 0$  and  $y_i \in \mathbb{Z}_2C$ . Then

$$xx^{\odot} = x_0^2(1 + y_1a + y_2b + y_3ab)(1 + y_1ae + y_2be + y_3abe) = 1$$

if and only if

$$\begin{cases} x_0^2(1 + y_1^2a^2e + y_2^2b^2 + y_3^2a^2b^2) = 1, \\ y_1(1+e) + y_2y_3b^2(1+e) = 0, \\ y_2(1+e) + y_1y_3a^2(1+e) = 0, \\ y_3(1+e) + y_1y_2(1+e) = 0. \end{cases} \quad (7)$$

Then from second and fourth equalitis we obtain

$$y_1(1+e) = y_1y_2^2b^2(1+e)$$

and

$$y_1(1+e)(1 + y_2^2b^2) = 0.$$

Since  $\chi(y_2) = 0$ , element  $1 + y_2^2b^2$  is a unit and from the equality above it follows that  $y_1(1+e) = 0$ . Clearly, (7) implies  $y_2(1+e) = y_3(1+e) = 0$ . By virtue of Proposition 2.7 [2] we have  $(1+e)y_i = (1+e)2y_i$  and  $y_i \in \mathbb{Z}_2C$ .

Clearly,  $y_i^2 = 0$  ( $i = 1, 2, 3$ ) and this together with (7) gives  $x_0^2 = 1$ . It is obvious that

$$y = 1 + y_1a + y_2b + y_3ab \quad (y_i \in (1 + e)\mathbb{Z}_2C)$$

is a central unit of order 2.

It is well known that

$$V(\mathbb{Z}_2C)[2] = C[2] \times T$$

and  $T \cap G = 1$ .

Clearly, we proved that  $V_\odot(\mathbb{Z}_2G)$  is generated by  $G$  and two central subgroups:

$$V(\mathbb{Z}_2C)[2] = \{x_0 \in V(\mathbb{Z}_2G) \mid x_0^2 = 1\},$$

$$W = \{1 + y_1a + y_2b + y_3ab \mid y_i \in (1 + e)\mathbb{Z}_2C\}.$$

Therefore

$$V_\odot(\mathbb{Z}_2G) = G \times T \times W.$$

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DEPARTMENT OF MATHEMATICS, BESSENYEI GYÖRGY TEACHERS' TRAINING COLLEGE,  
NYÍREGYHÁZA, HUNGARY